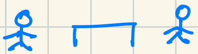
combinatorics

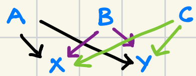
Introduction to Combinatorics

* branch of Math that is the science of counting
* often associated with probability

Eg.



3 choices → main dish
2 choices → dessert



$$\begin{pmatrix} AX & BX & CX \\ AY & BY & CY \end{pmatrix}$$

Total 6 combinations

lets say,

5 choices → main dish
4 choices → dessert

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 4 & 4 & 4 & 4 & 4 \end{array}$$

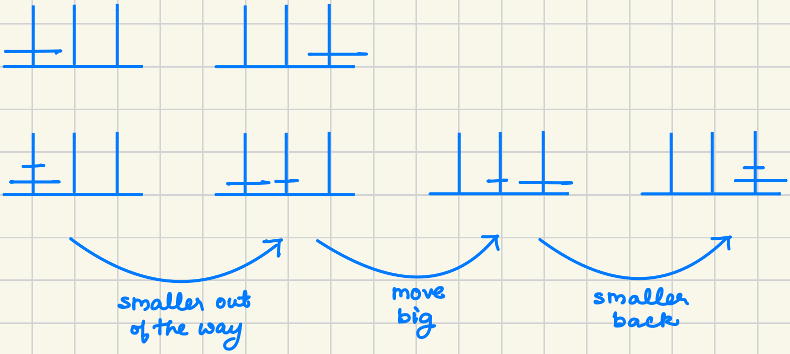
Total 20 combinations

→ If there are x choices for the first event and y choices for the second event, there are xy combinations of the two events

LETS BEGIN

Tower of Hanoi (Eduardo Lucas)

Game - Move all disks to another peg by moving only one disk at a time. Larger disk cannot be on top of smaller disk at any time.



$0 + 1 + 0 = 1$

$1 + 1 + 1 = 3$

$\vdots$

$3 + 1 + 3 = 7$

$7 + 1 + 7 = 15$

$15 + 1 + 15 = 31$

- Closed form formula,
  n disks, $M(n) = 2^n - 1$

- Recurrence relation,
  $$M(n) = \underbrace{M(n-1)}_{\text{smaller out of the way}} + \underbrace{1}_{\text{move big}} + \underbrace{M(n-1)}_{\text{smaller back}} = 2M(n-1) + 1$$

- Proving that both formulas match using Induction.

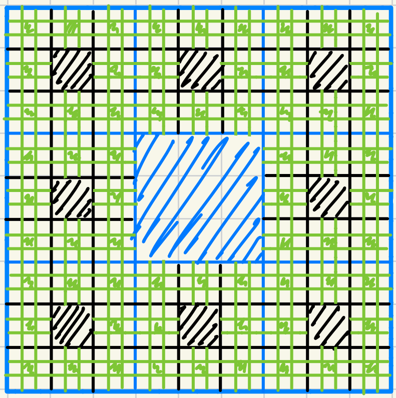
Base Case : $M(1) = 1$

Assume true : $M(n-1) = 2^{n-1} - 1$

$$
\begin{aligned}
M(n) &= 2M(n-1) + 1 \\
&= 2(2^{n-1} - 1) + 1 \\
&= 2 \cdot 2^{n-1} - 2 + 1 \\
&= 2 \cdot 2^{n-1} - 1 \\
&= 2^{\cancel{1}+n-\cancel{1}} - 1 \\
&= 2^n - 1
\end{aligned}
$$

$\therefore$ True

Sierpensky Carpet



1. Start with a square
2. Divide into 3×3 grid (9 smaller squares)
3. Remove central square
4. Repeat process for remaining 8 squares

Fractal Dimension ≈ 1.8128 (since we keep on removing center square)

Area $= 0$

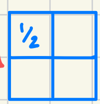
↳ But total length of all boundaries of squares ~ infinite

Scaling factor, $\frac{1}{r} = \frac{1}{3}$

no. of pieces, $n = 8$

dimension, D

$\boxed{r^D = n}$



$2^2 = 4$

So for Sierpinski :

$3^D = 8 \longrightarrow$ from step 1.

$\log_3(8) = D$

★ $D \approx 1.8927...$

Also,

A_0 = Area of original square

$A_1 = \frac{8}{9} A_0 \longrightarrow$ from step 1.

$A_2 = \left(\frac{8}{9}\right)^2 A_0$

. . .

$A_n = \left(\frac{8}{9}\right)^n \cdot A_0$

★ $\lim_{n \to \infty} \left(\frac{8}{9}\right)^n A_0 = 0$

→ The Floating Hanoi Towers (Gray Code) → only one bit changes at a time

- Hanoi towers broke off from land and float
- they balance slowly
- disks are heavy : 1, 2, 4, 8, . . . : eg. 8 > 1+2+4.
- any big move topples tower into sea so exchange sets which differ by 1

★ 1

2 > 1

4 > 1,2

8 > 1,2,4

16 > 1,2,4,8

The difference

normal	floating
• minimize moves	• maintain balance
• size constraint	• weight (2^n)
• recursion	• recursive balance

Variations:

- magnetic tower of hanoi
- revés puzzle

Cantor Set

$

1-D version of Sierpinski carpet

- How to build Cantor Sets:

1) Start with a solid line from 0 to 1

2) Delete the middle third (b/w $1/3$ and $2/3$). 2 segments left.

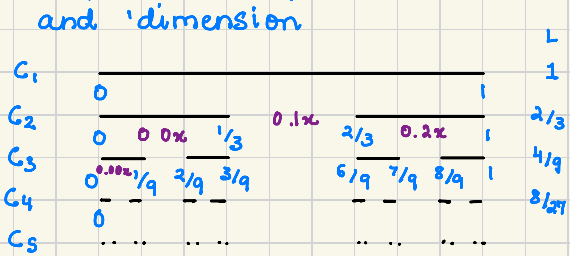
3) Delete the middle third of those segments

4) Repeat process forever. left with "Dust"

removed all 'length' but infinite points left.

★ seems obvious that we have infinite points, but all points add up to 0 length. (uncountably infinite with total length 0)

★ helped come up with better definitions for things like 'measure' and 'dimension

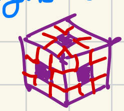


- length remaining after n steps: $L_n = \left(\frac{2}{3}\right)^n$

- total length: $\lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0$

- C consists precisely of the real numbers in $[0,1]$ whose 'base-3' expansions only contain digits 0 and 2.

→ 3D - Menger Sponge



Cube - 27 pieces - remove center and center face

1 + 6 = 7

Homogeneous Coordinates

topic of projective geometry